Package ‘weyl’

January 21, 2022

Type Package

Title The Weyl Algebra

Version 0.0-1

Depends spray (>= 1.0-19), methods, R (>= 3.5.0)

Maintainer Robin K. S. Hankin <hankin.robin@gmail.com>

Description A suite of routines for Weyl algebras. Notation follows
Coutinho (1995, ISBN 0-521-55119-6, ``A Primer of Algebraic D-Modules'').

License GPL (>= 2)

LazyData yes

Suggests knitr,rmarkdown,testthat

VignetteBuilder knitr

Imports mathjaxr, disordR (>= 0.0-8), freealg (>= 1.0-4)

URL https://github.com/RobinHankin/weyl

BugReports https://github.com/RobinHankin/weyl/issues

RdMacros mathjaxr

R topics documented:

`weyl-package` *The Weyl Algebra*

Description

A suite of routines for Weyl algebras. Notation follows Coutinho (1995, ISBN 0-521-55119-6, "A Primer of Algebraic D-Modules").

Details

The DESCRIPTION file:

Package:	weyl
Type:	Package
Title:	The Weyl Algebra
Version:	0.0-1
Depends:	spray (>= 1.0-19), methods, R (>= 3.5.0)
Authors@R:	person(given=c("Robin", "K. S."), family="Hankin", role = c("aut","cre"), email="hankin.robin@gmai
Maintainer:	Robin K. S. Hankin <hankin.robin@gmail.com>
Description:	A suite of routines for Weyl algebras. Notation follows Coutinho (1995, ISBN 0-521-55119-6, "A Prir
License:	GPL (>= 2)
LazyData:	yes
Suggests:	knitr,rmarkdown,testthat
VignetteBuilder:	knitr
Imports:	mathjaxr, disordR (>= 0.0-8), freealg (>= 1.0-4)
URL:	https://github.com/RobinHankin/weyl
BugReports:	https://github.com/RobinHankin/weyl/issues
RdMacros:	mathjaxr
Author:	Robin K. S. Hankin [aut, cre] (<https://orcid.org/0000-0001-5982-0415>)

Index of help topics:

```
Ops                     Arithmetic Ops Group Methods for the Weyl
                        algebra
coeffs                  Manipulate the coefficients of a weyl object
constant                The constant term
degree                  The degree of a 'weyl' object
derivation              Derivations
dim                     The dimension of a 'weyl' object
dot-class               Class "dot"
drop                    Drop redundant information
grade                   The grade of a weyl object
identity                The identity operator
print.weyl              Print methods for weyl objects
rweyl                   Random weyl objects
weyl                    The algebra and weyl objects
weyl-class              Class "weyl"
weyl-package            The Weyl Algebra
x_and_d                 Generating elements for the first Weyl algebra
```

zero	The zero operator

Author(s)

NA

Maintainer: Robin K. S. Hankin <hankin.robin@gmail.com>

Examples

```
x <- rweyl(d=1)
y <- rweyl(d=1)
z <- rweyl(d=1)

is.zero(x*(y*z) - (x*y)*z)  # should be TRUE
```

---

coeffs — *Manipulate the coefficients of a weyl object*

Description

Manipulate the coefficients of a weyl object. The coefficients are `disord` objects.

Usage

```
coeffs(S) <- value
```

Arguments

S	A weyl object
value	Numeric

Details

To access coefficients of a weyl object `S`, use `spray::coeffs(S)` [package idiom is `coeffs(S)`]. Similarly to access the index matrix use `index(s)`.

The replacement method is package-specific; use `coeffs(S) <-value`.

Value

Extraction methods return a `disord` object (possibly `dropped`); replacement methods return a `weyl` object.

Author(s)

Robin K. S. Hankin

Examples

```
a <- rweyl()
coeffs(a)
coeffs(a)[coeffs(a)<3] <- 100
```

`constant` *The constant term*

Description

The *constant* of a `weyl` object is the coefficient of the term with all zeros.

Usage

```
constant(x, drop = TRUE)
constant(x) <-  value
```

Arguments

`x`	Object of class `weyl`
`drop`	Boolean with default `TRUE` meaning to return the value of the coefficient, and `FALSE` meaning to return the corresponding Weyl object
`value`	Constant value to replace existing one

Value

Returns a numeric or weyl object

Note

The `constant.weyl()` function is somewhat awkward because it has to deal with the difficult case where the constant is zero and `drop=FALSE`.

Author(s)

Robin K. S. Hankin

Examples

```
a <- rweyl()+5
constant(a)
constant(a,drop=FALSE)

constant(a) <- 0
constant(a)
constant(a,drop=FALSE)

constant(a+66) == constant(a) + 66
```

`degree`	*The degree of a* `weyl` *object*

Description

The *degree* of a monomial weyl object $x^a\partial^b$ is defined as $a+b$. The degree of a general weyl object expressed as a linear combination of monimials is the maximum of the degrees of these monomials. Following Coutinho we have:

- $\deg(d_1 + d_2) \leq \max(\deg(d_1) + \deg(d_2))$
- $\deg(d_1 d_2) = \deg(d_1) + \deg(d_2)$
- $\deg(d_1 d_2 - d_2 d_1) \leq \deg(d_1) + \deg(d_2) - 2$

Usage

```
deg(S)
```

Arguments

`S`	Object of class `weyl`

Value

Nonnegative integer (or $-\infty$ for the zero Weyl object)

Note

The degree of the zero object is conventionally $-\infty$.

Author(s)

Robin K. S. Hankin

Examples

```
d1 <- rweyl(n=2)
d2 <- rweyl(n=2)

deg(d1+d2) <= deg(d1) + deg(d2)
deg(d1*d2) == deg(d1) + deg(d2)
deg(d1*d2-d2*d1) <= deg(d1) + deg(d2) -2
```

derivation *Derivations*

Description

A *derivation* D of an algebra A is a linear operator that satisfies $D(d_1 d_2) = d_1 D(d_2) + D(d_1)d_2$, for every $d_1, d_2 \in A$. If a derivation is of the form $D(d) = [d, f] = df - fd$ for some fixed $f \in A$, we say that D is an *inner* derivation.

Function `as.der()` returns a derivation with `as.der(f)(g)=fg-gf`.

Usage

```
as.der(S)
```

Arguments

`S`	Weyl object

Value

Returns a function, a derivation

Author(s)

Robin K. S. Hankin

Examples

```
o <- rweyl(n=2,d=2)
f <- as.der(o)

d1 <-rweyl(n=1,d=2)
d2 <-rweyl(n=2,d=2)

f(d1*d2) == d1*f(d2) + f(d1)*d2 # should be TRUE
```

dim *The dimension of a* `weyl` *object*

Description

The *dimension* of a weyl algebra is the number of variables needed; it is half the `spray::arity()`. The *dimension* of a Weyl algebra generated by $\{x_1, x_2, \ldots, x_n, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$ is n. It is the number of variables needed for the operators; it is half the `spray::arity()`.

Usage

```
## S3 method for class 'weyl'
dim(x)
```

Arguments

`x`	Object of class `weyl`

Value

Integer

Note

Empty spray objects give zero-dimensional weyl objects.

Author(s)

Robin K. S. Hankin

Examples

```
dim(rweyl())
```

`dot-class` *Class "dot"*

Description

The dot object is defined in the **freealg** package, and imported here, so that idiom like `.[x,y]` returns the commutator, that is, `xy-yx`.

Arguments

`x`	Object of any class
`i,j`	elements to commute
`...`	Further arguments to `dot_error()`, currently ignored

Value

Always returns an object of the same class as `xy`.

Author(s)

Robin K. S. Hankin

Examples

```
x <- rweyl(n=1,d=2)
y <- rweyl(n=1,d=2)
z <- rweyl(n=1,d=2)

.[x,.[y,z]] + .[y,.[z,x]] + .[z,.[x,y]]  # Jacobi identity
```

`drop` *Drop redundant information*

Description

Coerce constant weyl objects to numeric

Usage

```
drop(x)
```

Arguments

`x`	Weyl object

Details

If its argument is a constant weyl object, coerce to numeric.

Value

Returns either a length-one numeric vector or its argument, a weyl object

Note

Many functions in the package take `drop` as an argument which, if `TRUE`, means that the function returns a `drop`ped value.

Author(s)

Robin K. S. Hankin

Examples

```
a <- rweyl() + 67
drop(a)

drop(idweyl(9))

drop(constant(a,drop=FALSE))
```

grade *The grade of a weyl object*

Description

The *grade* of a homogenous term of a Weyl algebra is the sum of the powers. Thus the grade of $4xy^2\partial_x^3\partial_y^4$ is $1+2+3+4=10$.

The functionality documented here closely follows the equivalent in the **clifford** package.

Coutinho calls this the *symbol map*.

Usage

```
grade(C, n, drop=TRUE)
grade(C,n) <- value
grades(x)
```

Arguments

`C,x`	Weyl object
`n`	Integer vector specifying grades to extract
`value`	Replacement value, a numeric vector
`drop`	Boolean, with default `TRUE` meaning to coerce a constant operator to numeric, and `FALSE` meaning not to

Details

Function `grades()` returns an (unordered) vector specifying the grades of the constituent terms. Function `grades<-()` allows idiom such as `grade(x,1:2) <-7` to operate as expected [here to set all coefficients of terms with grades 1 or 2 to value 7].

Function `grade(C,n)` returns a Weyl object with just the elements of grade `g`, where `g %in% n`.

The zero grade term, `grade(C,0)`, is given more naturally by `constant(C)`.

Value

Integer vector or weyl object

Author(s)

Robin K. S. Hankin

Examples

```
a <- rweyl(30)

grades(a)
grade(a,1:4)
grade(a,5:9) <- -99
```

`identity` *The identity operator*

Description

The identity operator maps any function to itself.

Usage

```
idweyl(d)
## S3 method for class 'weyl'
as.id(S)
is.id(S)
```

Arguments

d	Integer specifying dimensionality of the weyl object (twice the spray arity)
S	A weyl object

Value

A weyl object corresponding to the identity operator

Note

The identity function cannot be called "`id()`" because then R would not know whether to create a `spray` or a `weyl` object.

Examples

```
idweyl(7)

a <- rweyl(d=5)
is.id(a)
is.id(1+a-a)

a == a*1
a == a*as.id(a)
```

`Ops` *Arithmetic Ops Group Methods for the Weyl algebra*

Description

Allows arithmetic operators to be used for spray calculations, such as addition, multiplication, division, integer powers, etc.

Idiom such as `x^2 + y*z/5` should work as expected. Operations are the same as those of the **spray** package except for `*`, which is interpreted as functional composition. A number of helper functions are documented here (which are not designed for the end-user).

Usage

```
## S3 method for class 'weyl'
Ops(e1, e2 = NULL)
weyl_prod_helper1(a,b,c,d)
weyl_prod_helper2(a,b,c,d)
weyl_prod_helper3(a,b,c,d)
weyl_prod_univariate_onerow(S1,S2,func)
weyl_prod_univariate_nrow(S1,S2)
weyl_prod_multivariate_onerow_singlecolumn(S1,S2,column)
weyl_prod_multivariate_onerow_allcolumns(S1,S2)
weyl_prod_multivariate_nrow_allcolumns(S1,S2)
weyl_power_scalar(S,n)
```

Arguments

`S,S1,S2,e1,e2`	Objects of class `weyl`, elements of a Weyl algebra
`a,b,c,d`	Integers, see details
`column`	column to be multiplied
`n`	Integer power (non-negative)
`func`	Function used for products

Details

All arithmetic is as for `spray` objects, apart from `*` and `^`. Here, `*` is interpreted as operator concatenation: Thus, if w_1 and w_2 are Weyl objects, then $w_1 w_2$ is defined as the operator that takes f to $w_1(w_2 f)$.

Functions such as `weyl_prod_multivariate_nrow_allcolumns()` are low-level helper functions with self-explanatory names. In this context, "`univariate`" means the first Weyl algebra, generated by $\{x, \partial\}$, subject to $x\partial - \partial x = 1$; and "`multivariate`" means the algebra generated by $\{x_1, x_2, \ldots, x_n, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$.

The product is somewhat user-customisable via option `prodfunc`, which affects function `weyl_prod_univariate_onero` Currently the package offers three examples: `weyl_prod_helper1()`, `weyl_prod_helper2()`, and `weyl_prod_helper3()`. These are algebraically identical but occupy different positions on the efficiency-readability scale. The option defaults to `weyl_prod_helper3()`, which is the fastest but most opaque. The vignette provides further details, motivation, and examples.

Value

Generally, return a weyl object

Note

Function `weyl_prod_univariate_nrow()` is present for completeness, it is not used in the package

Author(s)

Robin K. S. Hankin

Examples

```
x <- rweyl(n=1,d=2)
y <- rweyl(n=1,d=2)
z <- rweyl(n=2,d=2)


x*(y+z) == x*y + x*z
is.zero(x*(y*z) - (x*y)*z)
```

`print.weyl` *Print methods for weyl objects*

Description

Printing methods for weyl objects follow those for the **spray** package, with some additional functionality.

Usage

```
## S3 method for class 'weyl'
print(x, ...)
```

Arguments

`x`	A weyl object
`...`	Further arguments, currently ignored

Details

Option `polyform` determines whether the object is to be printed in matrix form or polynomial form: as in the **spray** package, this option governs dispatch to either `print_spray_polyform()` or `print_spray_matrixform()`.

Option `weylvars` controls the variable names by changing the `sprayvars` option which is used in the **spray** package. If `NULL` (the default), then sensible values are used: either `[xyz]` if the dimension is three or less, or integers.

If the user sets `weylvars`, the print method tries to do the Right Thing (tm). If set to `c("a","b","c")`, for example, the generators are named `c(" a"," b"," c","da","db","dc")` [note the spaces]. If the algebra is univariate, the names will be something like `d` and `x`. No checking is performed and if the length is not equal to the dimension, undesirable behaviour may occur. For the love of God, do not use a variable named `d`.

Note that, as for the **spray** package, this option has no algebraic significance: it only affects the print method.

Value

Returns a `weyl` object.

Author(s)

Robin K. S. Hankin

Examples

```
a <- rweyl()
print(a)
options(polyform=TRUE)
print(a)
```

`rweyl` *Random weyl objects*

Description

Creates random weyl objects: quick-and-dirty examples of Weyl algebra elements

Usage

```
rweyl(nterms = 3, vals = seq_len(nterms), dim = 3, powers = 0:2)
```

Arguments

`nterms`	Number of terms in output
`vals`	Values of coefficients
`dim`	Dimension of weyl object
`powers`	Set from which to sample the entries of the index matrix

Value

Returns a weyl object

Author(s)

Robin K. S. Hankin

Examples

```
rweyl()

rweyl(d=7)
```

`weyl` *The algebra and weyl objects*

Description

Basic functions for weyl objects

Usage

```
weyl(M)
is.weyl(M)
as.weyl(val,d)
is.ok.weyl(M)
```

Arguments

`M`	A weyl or spray object
`val,d`	Value and dimension for weyl object

Details

Function `weyl()` is the formal creator method; `is.weyl()` tests for weyl objects and `is.ok.weyl()` checks for well-formed sprays. Function `as.weyl()` tries (but not very hard) to infer what the user intended and return the right thing.

Value

Return a weyl or a Boolean

Author(s)

Robin K. S. Hankin

Examples

```
weyl(spray(matrix(1:36,6,6),1:6))

as.weyl(15,d=3)
```

`weyl-class` *Class "weyl"*

Description

The formal S4 class for weyls.

Objects from the Class

Objects *can* be created by calls of the form `new("weyl",...)` but this is not encouraged. Use functions `weyl()` or `as.weyl()` instead.

Author(s)

Robin K. S. Hankin

`x_and_d` *Generating elements for the first Weyl algebra*

Description

Variables `x` and `d` correspond to operator x and ∂_x; they are provided for convenience. These elements generate the one-dimensional Weyl algebra.

Note that a similar system for multivariate Weyl algebras is not desirable. We might want to consider the Weyl algebra generated by $\{x, y, z, \partial_x, \partial_y, \partial_z\}$ and correspondingly define R variables `x,y,z,dx,dy,dz`. But then variable `x` is ambiguous: is it a member of the first Weyl algebra or the third?

Usage

```
data(x_and_d)
```

Author(s)

Robin K. S. Hankin

Examples

```
d*x-x*d

(1-d*x*d)*(x^2-d^3)
```

`zero` *The zero operator*

Description

The zero operator maps any function to the zero function (which maps anything to zero). To test for being zero, use `spray::is.zero()`; package idiom would be `is.zero()`.

Usage

```
zero(d)
```

Arguments

`d`	Integer specifying dimensionality of the weyl object (twice the spray arity)

Value

A weyl object corresponding to the zero operator (or a Boolean for `is.zero()`)

Examples

```
a <- rweyl(d=5)
is.zero(a)
is.zero(a-a)
is.zero(a*0)

a == a + zero(dim(a))
```

Index