

Proposal for optimal control GUI project

1 Set-up

Suppose we're given a vector N noisy mosquito population estimates $\mathbf{y} = (y_1, y_2, \dots, y_N)$ based on trap measurements which occur at times (t_1, t_2, \dots, t_N) , representative of the population's evolution over the course of an entire season or year τ .

Problem: Use the population data to determine a schedule for N_p impulses to occur over a year, assuming that each impulse reduces the vector population by a fraction ρ . Let $\mathbf{z} = (z_1, z_2, \dots, z_p)$ denote the timing of the N_p impulses.

Solution: Assume that the true population obeys the following dynamics:

$$\dot{x}(t) = \Lambda(t) - \mu x(t), \quad (1)$$

where $x(t)$ is the vector population at time t , $\Lambda(t)$ is a time-varying emergence rate, and $1/\mu$ is the natural vector lifetime.

First, we fit this model to our trap data assuming we have an initial reasonable guess for μ based on the climate of the area under consideration. Specifically, we will attempt to find a τ -periodic $\Lambda(t)$ for which the corresponding τ -periodic solution $x(t)$ to Eq. (1) most closely matches the measurement data \mathbf{y} over a single period. If the resulting fit is unsatisfactory, other reasonable values of μ can be tested to see if a better fit can be achieved, or we can try to include μ as additional parameter to fit. A simple scheme for actually doing such a fitting, assuming a value for μ is outlined in the next section.

Next, we calculate the Fourier modes of our fitted emergence rate $\Lambda(t)$, and we denote the n^{th} mode Λ_n . When the N_p impulses are applied periodically with period τ to the dynamics in Eq. (1) under the fitted $\Lambda(t)$, the average $\langle x \rangle$ of the corresponding periodic population trajectory is given by

$$\langle x \rangle = \sum_{j,k=-\infty}^{\infty} \left[\sum_{\substack{\sum_{i=1}^p n_i=k \\ \sum_{i=1}^p m_i=k+j}} \left(\frac{\prod_{i=1}^{N_p} P_{n_i} Q_{-m_i} e^{\frac{-2\pi i(n_i - m_i)z_i}{\tau}}}{1 - N_p \frac{\ln(1-\rho)}{\mu\tau} - \frac{2\pi ik}{\mu\tau}} \right) \frac{\Lambda_j}{\mu} \right], \quad (2)$$

where

$$\begin{aligned} P_n &= \rho \frac{1}{\ln \left[\frac{1}{1-\rho} \right] - 2\pi in} \\ Q_n &= \frac{\rho}{1-\rho} \frac{1}{\ln \left[\frac{1}{1-\rho} \right] + 2\pi in}. \end{aligned} \tag{3}$$

and where the expression

$$\sum_{\substack{\sum_{l=1}^p n_l = k \\ \sum_{l=1}^p m_l = k+j}} \tag{4}$$

denotes a summation over all integer p -tuples (n_1, n_2, \dots, n_p) and (m_1, m_2, \dots, m_p) such that $\sum_{l=1}^p n_l = k$ and $\sum_{l=1}^p m_l = k + j$.

To solve our optimal control problem, we simply find the global minimum of $\langle x \rangle$ with respect to the impulse timings \mathbf{z} . This will give us the optimal schedule for P impulses given the time-dependent emergence rate as estimated from our data. Although we have the analytic expression for $\langle x \rangle$, finding the global minimum may be challenging due to the large number of terms being summed over and the oscillatory behavior of these terms as a function of \mathbf{z} . Hopefully, these difficulties can be somewhat mitigated by the decay of P_n and Q_n as $1/|n|$ (which limits the number of terms that need to be calculated in practice), and perhaps a good initial guess for \mathbf{z} which will allow use of simpler local optimization methods like Newton's method for finding the zeros of $\partial_{\mathbf{z}} \langle x \rangle$.

Input / Output: The user will input the population data \mathbf{y} , corresponding times (t_1, t_2, \dots, t_N) , season length τ , estimated vector lifetime $1/\mu$, percent knockdown ρ , and number of impulses P . The output will display the population curve from the data (assuming a linear interpolation between points), the fitted uncontrolled population curve, the optimal schedule for impulse timings, and the corresponding population curve under the impulse controls. If the algorithm doesn't take too long, we could also perform the optimization for $P = 1, 2, \dots, P_{max}$ and plot out the percent population reduction as a function of the number of impulses P . It would be interesting to see a what point we get diminishing returns when adding more impulses.

Caveat: In the problem set-up, we are looking for an impulse protocol to apply over a single season of length τ which will maximally reduce the mosquito population, given an emergence rate $\Lambda(t)$ inferred from our population model and single-season data. We are actually finding a τ -periodic impulse protocol which minimizes, on average, the periodic equilibrium population, given a τ -periodic emergence rate $\Lambda(t)$ inferred from our population model and single-season data. I think these problems will be roughly equivalent if we include enough zero measurements around the seasonal peak in the data vector \mathbf{y} so that if we were to stitch two such data sequences together, one after the other, the mosquito population will have effectively died out between the peaks (I'm still thinking about the logic of this, but I think it's right). By including these zeros, when stitching together copies of a single-season population curve to form a periodic population curve, the populations between seasons will effectively evolve independently of one another, and

there should be no interactions or synergistic effects between control impulses from different seasons. I think if this is the case, the optimal periodic control protocol will also optimally reduce the population over a single season.

2 Simple scheme for model fitting

Here, I'm assume that we only have one season of data to work with. If we end up using the scheme outline here, then we'll probably want to find a way to incorporate multiple seasons of data.

We don't really know anything about $\Lambda(t)$ and need to make some assumptions, so we are going to assume its form is a linear interpolation of the values $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$ at times (t_1, t_2, \dots, t_N) :

$$\Lambda(t) = \lambda_{k+1} \frac{t - t_k}{\Delta t_k} + \lambda_k \frac{t_{k+1} - t}{\Delta t_k}, \quad t \in [t_k, t_{k+1}], \quad k = 1, 2, \dots, N, \quad (5)$$

where $\Delta t_k = t_{k+1} - t_k$. Note that the interval $[t_N, t_{N+1}]$ represents the interval between the last data point of the first season and the first data point of the second season, $\lambda_{k+N} = \lambda_k$ by periodicity, and we have the identity $\tau = \sum_{k=1}^N \Delta t_k$. Under this form of $\Lambda(t)$, the dynamics in Eq. (1) have a closed form solution $x(t)$, and we can find analytic expressions for the Fourier modes Λ_n . The Fourier modes are not needed for the curve fitting, but are included at the end of this document for reference. Denoting the values of $x(t)$ at times (t_1, t_2, \dots, t_N) by $\mathbf{x} = (x_1, x_2, \dots, x_N)$, and where $x_{k+N} = x_k$ for periodic $x(t)$, we find the following relationship between \mathbf{x} and $\boldsymbol{\lambda}$:

$$x_{k+1} = x_k e^{\mu \Delta t_k} + \left[\frac{1 - e^{-\mu \Delta t_k}}{\mu \Delta t_k} - e^{-\mu \Delta t_k} \right] \frac{\lambda_k}{\mu} + \left[1 - \frac{1 - e^{-\mu \Delta t_k}}{\mu \Delta t_k} \right] \frac{\lambda_{k+1}}{\mu} \quad (6)$$

In matrix form, we have

$$J\mathbf{x} = M \frac{\boldsymbol{\lambda}}{\mu} \quad (7)$$

where J is the following $N \times N$ invertible matrix:

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & -e^{-\mu \Delta t_N} \\ -e^{-\mu \Delta t_1} & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -e^{-\mu \Delta t_2} & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & -e^{-\mu \Delta t_3} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -e^{-\mu \Delta t_{N-1}} & 1 \end{pmatrix}, \quad (8)$$

and M is the following $N \times N$ invertible matrix:

$$M = \begin{pmatrix} b_N & 0 & 0 & 0 & \dots & 0 & a_N \\ a_1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_2 & b_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_3 & b_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & b_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \dots & a_{N-1} & b_{N-1} \end{pmatrix}, \quad (9)$$

where

$$a_k = \frac{1 - e^{-\mu\Delta t_k}}{\mu\Delta t_k} - e^{-\mu\Delta t_k}, \quad (10)$$

$$b_k = 1 - \frac{1 - e^{-\mu\Delta t_k}}{\mu\Delta t_k}. \quad (11)$$

We thus find a one-to-one correspondence between discretized trajectories \mathbf{x} and discretized emergence rates $\boldsymbol{\lambda}$ according to $\boldsymbol{\lambda} = \mu M^{-1} J \mathbf{x}$. If \mathbf{x} is a feasible trajectory, meaning that it is a discretization of a solution to Eq. (1) for some linear interpolated $\Lambda(t)$, then vector $\boldsymbol{\lambda} = \mu M^{-1} J \mathbf{x}$ will have all non-negative components (negative values for the emergence rates have no biological meaning). Given any N dimensional vector of non-negative numbers \mathbf{v} , if $\mu M^{-1} J \mathbf{v}$ has negative components, then \mathbf{v} is a discretization of a trajectory which only approximately obeys Eq. (1) for non-negative linearly interpolated $\Lambda(t)$'s.

Consider now the vector of noisy observations over a single season $\mathbf{y} = (y_1, y_2, \dots, y_N)$. We don't know the actual trajectory \mathbf{x} , so we use \mathbf{y} as an estimator for \mathbf{x} . If \mathbf{y} is itself a feasible trajectory, then we can compute $\boldsymbol{\lambda} = M^{-1} K \mathbf{y}$ to find a non-negative $\boldsymbol{\lambda}$ whose interpolation $\Lambda(t)$, when applied to Eq. (1), gives a trajectory $x(t)$ which interpolates exactly through the data \mathbf{y} . If this is the case, then we have the needed emergence rate $\Lambda(t)$, and we can proceed with computing optimal control protocols using its Fourier modes. In all likelihood, due to noise and model imperfections, \mathbf{y} will not be a feasible trajectory, and the equation $K^{-1} M \boldsymbol{\lambda} = \mathbf{y}$ will have no non-negative solutions $\boldsymbol{\lambda}$. In this case, finding the best possible $\boldsymbol{\lambda}$ (in the mean square sense) can be formulated as a constrained optimization problem:

$$\boldsymbol{\lambda} = \underset{\hat{\boldsymbol{\lambda}}}{\operatorname{argmin}} \left\{ \|K^{-1} M \hat{\boldsymbol{\lambda}} - \mathbf{y}\|^2 \mid \hat{\lambda}_i \geq 0 \right\}. \quad (12)$$

This is a convex quadratic optimization problem with solutions constrained to the non-negative orthant. There are packages in Mathematica and Matlab for efficiently solving such problems. After finding $\boldsymbol{\lambda}$, the corresponding feasible trajectory $\hat{\mathbf{y}}$ can be calculated from $\hat{\mathbf{y}} = K^{-1} M \boldsymbol{\lambda}$. This $\hat{\mathbf{y}}$ will be as close as possible to \mathbf{y} in the mean square sense, and we will have $\hat{\mathbf{y}} = \mathbf{y}$ only if \mathbf{y} is feasible. Some examples of the fitting procedure are shown below. For these examples, our fitting procedure yields feasible trajectories which match the data fairly well.

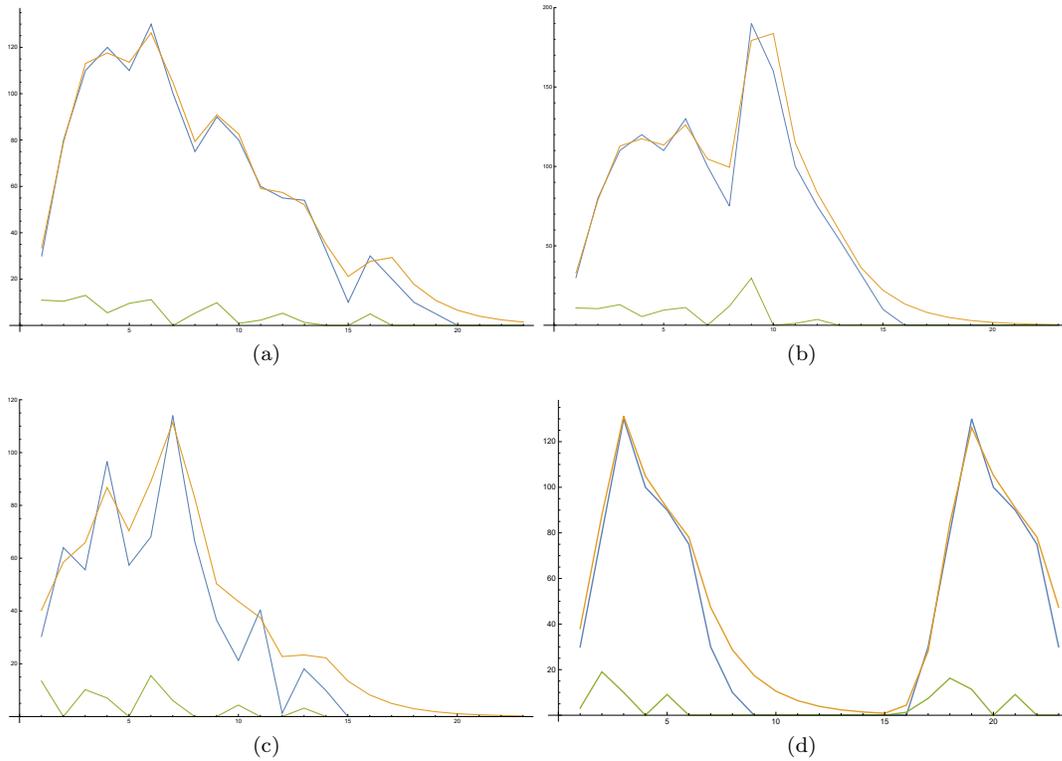


Figure 1: Sample plots showing the fitting procedure. 23 points were generated sort of randomly by me for each plot, representing four noisy data sets \mathbf{y} . These points are linearly interpolated and shown in blue. The green curves give the linearly interpolated λ as calculated from Eq. (12), and the corresponding closest feasible trajectory $\hat{\mathbf{y}}$ is interpolated and shown in orange.

3 Fourier modes for linearly interpolated $\Lambda(t)$

Given the discretized emergence rate $(\lambda_1, \lambda_2, \dots, \lambda_N)$ at times (t_1, t_2, \dots, t_N) and assuming a linear interpolation, the 0^{th} and n^{th} Fourier modes of $\Lambda(t)$ are respectively given by

$$\Lambda_0 = \frac{1}{\tau} \sum_{k=1}^N \frac{\lambda_k + \lambda_{k+1}}{2} \Delta t_k, \quad (13)$$

$$\begin{aligned} \Lambda_n = & \sum_{k=1}^N \frac{\lambda_k}{2\pi i n} e^{-\frac{2\pi i n t_k}{\tau}} \left[1 - \frac{\tau}{2\pi i n} \left(1 - e^{-\frac{2\pi i n \Delta t_k}{\tau}} \right) \right] \dots \\ & + \frac{\lambda_{k+1}}{2\pi i n} e^{-\frac{2\pi i n t_k}{\tau}} \left[\frac{\tau}{2\pi i n} \left(1 - e^{-\frac{2\pi i n \Delta t_k}{\tau}} \right) - e^{-\frac{2\pi i n \Delta t_k}{\tau}} \right]. \end{aligned} \quad (14)$$