

# kKT Conditions for Zero-Inflated Regression

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Some technical details on the solution path are presented, supplementary to the publications [Wang et al. \(2014, 2015\)](#).

## 1 Zero-inflated Poisson regression

Assume response variable  $Y$  has a zero-inflated Poisson distribution, denote  $d_1 + 1$  and  $d_2 + 1$ -dimensional vectors  $B_i$  and  $G_i$ . The parameters  $\mu = (\mu_1, \mu_2, \dots, \mu_n)^\top$  and  $\pi_i$  are modeled with  $\log(\mu_i) = B_i^\top \beta$  and  $\text{logit}(\pi_i) = \log(\pi_i/(1 - \pi_i)) = G_i^\top \zeta$  for covariate matrix  $B$  and  $G$ , which can be different. To include an intercept, let  $B_{i0} = G_{i0} = 1$ . Define  $\Phi = (\zeta^\top, \beta^\top)^\top$  with length  $d = d_1 + d_2 + 2$ . The log-likelihood function  $\ell_{ZIP}(\Phi; y)$  is given by

$$\begin{aligned} \ell_{ZIP}(\Phi; y) = & \sum_{y_i=0} \log(\exp(G_i^\top \zeta) + \exp(-e^{B_i^\top \beta})) + \sum_{y_i>0} (y_i B_i^\top \beta - \exp(B_i^\top \beta)) \\ & - \sum_{i=1}^n \log(1 + \exp(G_i^\top \zeta)) - \sum_{y_i>0} \log(y_i!). \end{aligned} \quad (1)$$

We define a penalized log-likelihood function for ZIP model:

$$p\ell_{ZIP}(\Phi; y) = \ell_{ZIP}(\Phi; y) - p(\zeta, \beta), \quad (2)$$

where

$$p(\zeta, \beta) = n \sum_{j=1}^{d_1} (\alpha_1 \lambda_1 |\zeta_j| + \frac{\lambda_1(1 - \alpha_1)}{2} \zeta_j^2) + n \sum_{k=1}^{d_2} (\alpha_2 \lambda_2 |\beta_k| + \frac{\lambda_2(1 - \alpha_2)}{2} \beta_k^2). \quad (3)$$

In the **R** package **mpath**,  $\alpha_1, \alpha_2$  are labeled as **alpha.zero**, **alpha.count**, respectively;  $\lambda_1, \lambda_2$  are labeled as **lambda.zero**, **lambda.count**, respectively. For an maximizer  $\hat{\Phi}$  of  $p\ell_{ZIP}(\Phi; y)$  if and only if the subdifferential of  $p\ell_{ZIP}(\Phi; y)$  at  $\hat{\Phi}$  is 0. Take derivatives:

$$\begin{aligned} \frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \zeta_j} &= \sum_{y_i=0} \frac{\exp(G_i^\top \zeta) G_{ij}}{\exp(G_i^\top \zeta) + \exp(-\exp(B_i^\top \beta))} - \sum_{i=1}^n \frac{\exp(G_i^\top \zeta) G_{ij}}{1 + \exp(G_i^\top \zeta)}, \\ \frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \beta_k} &= \sum_{y_i=0} \frac{\exp(-\exp(B_i^\top \beta))(-\exp(B_i^\top \beta)) B_{ik}}{\exp(G_i^\top \zeta) + \exp(-\exp(B_i^\top \beta))} \\ &\quad + \sum_{y_i>0} (y_i B_{ik} - \exp(B_i^\top \beta) B_{ik}). \end{aligned} \quad (4)$$

In this document, we take  $j = 1, \dots, d_1, k = 1, \dots, d_2$  unless otherwise specified. The subdifferential of  $p(\zeta, \beta)$  at  $\hat{\zeta}_j \neq 0$  is  $n\alpha_1\lambda_1\text{sign}(\zeta_j) + \lambda_1(1 - \alpha_1)\zeta_j$ , and the subdifferential of  $p(\zeta, \beta)$  at  $\hat{\beta}_k \neq 0$  is  $n\alpha_2\lambda_2\text{sign}(\beta_k) + \lambda_2(1 - \alpha_2)\beta_k$ . The subdifferential of  $p(\zeta, \beta)$  at  $\hat{\zeta}_j = 0$  is  $n\alpha_1\lambda_1 e_1$  for  $e_1 \in [-1, 1]$ , and the subdifferential of  $p(\zeta, \beta)$  at  $\hat{\beta}_k = 0$  is  $n\alpha_2\lambda_2 e_2$  for  $e_2 \in [-1, 1]$ . Together, we have the KKT conditions for an maximizer  $\hat{\Phi}$  of  $p\ell_{ZIP}(\Phi; y)$ :

$$\begin{aligned} \text{if } \hat{\zeta}_j \neq 0 : & -\frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \zeta_j} + n\alpha_1\lambda_1\text{sign}(\zeta_j) + \lambda_1(1 - \alpha_1)\zeta_j = 0, \\ \text{if } \hat{\zeta}_j = 0 : & -\frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \zeta_j} + n\alpha_1\lambda_1 e_1 = 0, \\ \text{if } \hat{\beta}_k \neq 0 : & -\frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \beta_k} + n\alpha_2\lambda_2\text{sign}(\beta_k) + \lambda_2(1 - \alpha_2)\beta_k = 0, \\ \text{if } \hat{\beta}_k = 0 : & -\frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \beta_k} + n\alpha_2\lambda_2 e_2 = 0. \end{aligned} \tag{5}$$

Therefore, for  $\hat{\zeta}_j = 0, \hat{\beta}_k = 0$ , it must be:

$$\left| \frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \zeta_j} \right| \leq n\alpha_1\lambda_1, \quad \left| \frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \beta_k} \right| \leq n\alpha_2\lambda_2. \tag{6}$$

Denote  $\lambda_{1,\max}$  and  $\lambda_{2,\max}$  the smallest values of  $\lambda_1$  and  $\lambda_2$ , respectively, such that  $\hat{\zeta}_j = 0, \hat{\beta}_k = 0$ , and  $(\lambda_{1,\max}, \lambda_{2,\max})$  can be determined by (6) and the following quantities:

$$\begin{aligned} \frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \zeta_j} &= \sum_{y_i=0} \frac{\exp(\zeta_0)G_{ij}}{\exp(\zeta_0) + \exp(-\exp(\beta_0))} - \sum_{i=1}^n \frac{\exp(\zeta_0)G_{ij}}{1 + \exp(\zeta_0)}, \\ \frac{\partial \ell_{ZIP}(\Phi; y)}{\partial \beta_k} &= \sum_{y_i=0} \frac{\exp(-\exp(\beta_0))(-\exp(\beta_0))B_{ik}}{\exp(\zeta_0) + \exp(-\exp(\beta_0))} \\ &\quad + \sum_{y_i>0} (y_i B_{ik} - \exp(\beta_0) B_{ik}). \end{aligned} \tag{7}$$

There is an alternative approach to construct  $(\lambda_{1,\max}, \lambda_{2,\max})$  as in Wang et al. (2014). In mixture models, the EM algorithm is set up by imposing missing data into the problem. Suppose we could observe which zeros came from the zero state and which came from Poisson state; i.e., suppose we knew  $z_i = 1$  when  $y_i$  is from zero state, and  $z_i = 0$  when  $y_i$  is from the Poisson state. Denote  $z = (z_1, z_2, \dots, z_n)^\top$ . The complete data  $(y, z)$  log-likelihood function can be written as

$$\begin{aligned} \ell_{ZIP}^c(\Phi; y, z) &= \sum_{i=1}^n \{z_i G_i^\top \zeta - \log(1 + \exp(G_i^\top \zeta))\} \\ &\quad + \sum_{i=1}^n (1 - z_i) \{y_i B_i^\top \beta - \exp(B_i^\top \beta) - \log(y_i!)\}. \end{aligned} \tag{8}$$

The complete data penalized log-likelihood function is then given by

$$\begin{aligned} p\ell_{ZIP}^c(\Phi; y, z) &= \sum_{i=1}^n \{z_i G_i^\top \zeta - \log(1 + \exp(G_i^\top \zeta))\} \\ &\quad + \sum_{i=1}^n (1 - z_i) \{y_i B_i^\top \beta - \exp(B_i^\top \beta) - \log(y_i!)\} - p(\zeta, \beta). \end{aligned} \quad (9)$$

Taking derivatives of (8), we obtain

$$\begin{aligned} \frac{\partial \ell_{ZIP}^c(\Phi; y, z)}{\partial \zeta_j} &= \sum_{i=1}^n \left\{ z_i G_{ij} - \frac{\exp(G_i^\top \zeta) G_{ij}}{1 + \exp(G_i^\top \zeta)} \right\}, \\ \frac{\partial \ell_{ZIP}^c(\Phi; y, z)}{\partial \beta_k} &= \sum_{i=1}^n (1 - z_i) \{y_i B_{ik} - \exp(B_i^\top \beta) B_{ik}\}. \end{aligned} \quad (10)$$

The KKT conditions of an maximizer  $\hat{\Phi}$  for  $p\ell_{ZIP}^c(\Phi; y, z)$  are given by:

$$\begin{aligned} \text{if } \hat{\zeta}_j \neq 0 : & -\frac{\partial \ell_{ZIP}^c(\Phi; y)}{\partial \zeta_j} + n\alpha_1 \lambda_1 \text{sign}(\zeta_j) + \lambda_1(1 - \alpha_1)\zeta_j = 0, \\ \text{if } \hat{\zeta}_j = 0 : & -\frac{\partial \ell_{ZIP}^c(\Phi; y)}{\partial \zeta_j} + n\alpha_1 \lambda_1 e_1 = 0, \\ \text{if } \hat{\beta}_k \neq 0 : & -\frac{\partial \ell_{ZIP}^c(\Phi; y)}{\partial \beta_k} + n\alpha_2 \lambda_2 \text{sign}(\beta_k) + \lambda_2(1 - \alpha_2)\beta_k = 0, \\ \text{if } \hat{\beta}_k = 0 : & -\frac{\partial \ell_{ZIP}^c(\Phi; y)}{\partial \beta_k} + n\alpha_2 \lambda_2 e_2 = 0. \end{aligned} \quad (11)$$

Therefore, for  $\hat{\zeta}_j = 0, \hat{\beta}_k = 0$ , it must be:

$$\left| \frac{\partial \ell_{ZIP}^c(\Phi; y)}{\partial \zeta_j} \right| \leq n\alpha_1 \lambda_1, \quad \left| \frac{\partial \ell_{ZIP}^c(\Phi; y)}{\partial \beta_k} \right| \leq n\alpha_2 \lambda_2. \quad (12)$$

The EM algorithm estimates  $z_i$  at iteration  $m$  by its conditional mean  $z_i^{(m)}$  given below:

$$z_i^{(m)} = \begin{cases} [1 + \exp(-G_i^\top \zeta^{(m)} - \exp(B_i^\top \beta^{(m)}))]^{-1}, & \text{if } y_i = 0, \\ 0, & \text{if } y_i = 1, 2, \dots \end{cases} \quad (13)$$

Let  $\zeta^{(m)} = \zeta, \beta^{(m)} = \beta$ , then (13) becomes

$$z_i = \begin{cases} [1 + \exp(-G_i^\top \zeta - \exp(B_i^\top \beta))]^{-1}, & \text{if } y_i = 0, \\ 0, & \text{if } y_i = 1, 2, \dots \end{cases} \quad (14)$$

It is a simple exercise to show that the right hand side of (10) is the same as that of (4) once (14) is plugged into (10). Hence, the KKT conditions (11) are the same as (5) once (14) is plugged into (10). These connections offer a different method (see Wang et al. (2014)) to derive  $(\hat{\lambda}_{1,\max}, \hat{\lambda}_{2,\max})$  such that  $\hat{\zeta}_j = 0, \hat{\beta}_k = 0$  hold. We first estimate  $\zeta_0, \beta_0$  for an intercept-only ZIP model, then (14) reduces to

$$z_i = \begin{cases} [1 + \exp(-\zeta_0 - \exp(\beta_0))]^{-1}, & \text{if } y_i = 0, \\ 0, & \text{if } y_i = 1, 2, \dots \end{cases} \quad (15)$$

Plugging in (15),  $(\hat{\lambda}_{1,\max}, \hat{\lambda}_{2,\max})$  are computed based on (12). Furthermore, we have shown that  $(\lambda_{1,\max}, \lambda_{2,\max}) = (\hat{\lambda}_{1,\max}, \hat{\lambda}_{2,\max})$  holds.

## 2 Zero-inflated negative binomial regression

Assume response variable  $Y$  has a zero-inflated negative binomial distribution, denote  $d_1 + 1$  and  $d_2 + 1$ -dimensional vectors  $B_i$  and  $G_i$ , respectively. As before, the first entry of these vectors is 1. In ZINB regression, assume  $\log(\mu_i) = B_i^\top \beta$  and  $\log(\frac{p_i}{1-p_i}) = G_i^\top \zeta$  where  $\zeta = (\zeta_0, \zeta_1, \dots, \zeta_{d_1})$  and  $\beta = (\beta_0, \beta_1, \dots, \beta_{d_2})$  are unknown parameters. Here  $\zeta_0$  and  $\beta_0$  are intercepts. For  $n$  independent random samples, denote  $\Phi = (\zeta^\top, \beta^\top, \theta)^\top$ , the log-likelihood function is then given by

$$\begin{aligned} \ell_{ZINB}(\Phi; y) = & \sum_{y_i=0} \log \left[ p_i + (1-p_i) \left( \frac{\theta}{\mu_i + \theta} \right)^\theta \right] \\ & + \sum_{y_i>0} \log \left[ (1-p_i) \frac{\Gamma(\theta + y_i)}{\Gamma(y_i + 1)\Gamma(\theta)} \left( \frac{\mu_i}{\mu_i + \theta} \right)^{y_i} \left( \frac{\theta}{\mu_i + \theta} \right)^\theta \right], \end{aligned}$$

where  $\mu_i = \exp(B_i^\top \beta)$  and  $p_i = \frac{\exp(G_i^\top \zeta)}{1 + \exp(G_i^\top \zeta)}$ . The derivatives are given by:

$$\begin{aligned} \frac{\partial \ell_{ZINB}(\Phi; y)}{\partial \zeta_j} = & \sum_{y_i=0} \frac{\frac{\partial p_i}{\partial \zeta_j} - \frac{\partial p_i}{\partial \zeta_j} \left( \frac{\theta}{\mu_i + \theta} \right)^\theta}{p_i + (1-p_i) \left( \frac{\theta}{\mu_i + \theta} \right)^\theta} - \sum_{y_i>0} \frac{\partial p_i}{\partial \zeta_j} \frac{1}{1-p_i}, \\ \frac{\partial \ell_{ZINB}(\Phi; y)}{\partial \beta_k} = & \sum_{y_i=0} \frac{- (1-p_i) \frac{\partial u_i}{\partial \beta_k} \frac{\theta \left( \frac{\theta}{\mu_i + \theta} \right)^\theta}{\mu_i + \theta}}{p_i + (1-p_i) \left( \frac{\theta}{\mu_i + \theta} \right)^\theta} + \sum_{y_i>0} \frac{\partial u_i}{\partial \beta_k} \left( \frac{y_i}{\mu_i} - \frac{y_i + \theta}{\mu_i + \theta} \right), \end{aligned} \quad (16)$$

where

$$\begin{aligned} \frac{\partial p_i}{\partial \zeta_j} &= \frac{G_{ij} \exp(G_i^\top \zeta)}{(1 + \exp(G_i^\top \zeta))^2}, \\ \frac{\partial u_i}{\partial \beta_k} &= B_{ik} \exp(B_i^\top \beta). \end{aligned}$$

For variable selection, consider a penalized ZINB model:

$$p\ell_{ZINB}(\Phi; y) = \ell(\Phi) - p(\zeta, \beta), \quad (17)$$

where  $p(\zeta, \beta)$  is given by (3). The KKT conditions for an maximizer  $\hat{\Phi}$  of  $p\ell_{ZINB}(\Phi)$  can be derived. Therefore, for  $\hat{\zeta}_j = 0, \hat{\beta}_k = 0$ , it must be:

$$\left| \frac{\partial \ell_{ZINB}(\Phi; y)}{\partial \zeta_j} \right| \leq n\alpha_1 \lambda_1, \quad \left| \frac{\partial \ell_{ZINB}(\Phi; y)}{\partial \beta_k} \right| \leq n\alpha_2 \lambda_2. \quad (18)$$

Denote  $\lambda_{1,\max}$  and  $\lambda_{2,\max}$  the smallest values of  $\lambda_1$  and  $\lambda_2$ , respectively, such that  $\hat{\zeta}_j = 0, \hat{\beta}_k = 0$ , and  $(\lambda_{1,\max}, \lambda_{2,\max})$  can be determined by (16), (18) and the following quantities:

$$p_i = \frac{\exp(\zeta_0)}{1 + \exp(\zeta_0)}, \quad \frac{\partial p_i}{\partial \zeta_j} = \frac{G_{ij} \exp(\zeta_0)}{(1 + \exp(\zeta_0))^2}, \quad \mu_i = \exp(\beta_0), \quad \frac{\partial u_i}{\partial \beta_k} = B_{ik} \exp(\beta_0). \quad (19)$$

Consider an EM algorithm to optimize (17). Let  $z_i = 1$  if  $y_i$  is from the zero state and  $z_i = 0$  if  $y_i$  is from the NB state. Since  $z = (z_1, \dots, z_n)^T$  is not observable, it is often treated as missing data. The EM algorithm is particularly attractive to missing data problems. If complete data  $(y, z)$  are available, the complete data log-likelihood function is given by

$$\ell_{ZINB}^c(\Phi; y) = \sum_{i=1}^n \{ (z_i G_i^T \zeta - \log(1 + \exp(G_i^T \zeta)) + (1 - z_i) \log(f(y_i; \beta, \theta))) \}, \quad (20)$$

and the complete data penalized log-likelihood function is given by

$$p\ell_{ZINB}^c(\Phi; y, z) = \sum_{i=1}^n \{ (z_i G_i^T \zeta - \log(1 + \exp(G_i^T \zeta)) + (1 - z_i) \log(f(y_i; \beta, \theta))) \} - p(\zeta, \beta),$$

where  $f(y_i; \beta, \theta) = \frac{\Gamma(\theta + y_i)}{\Gamma(y_i + 1)\Gamma(\theta)} \left(\frac{\mu_i}{\mu_i + \theta}\right)^{y_i} \left(\frac{\theta}{\mu_i + \theta}\right)^\theta$  and  $\mu_i = \exp(B_i^T \beta)$ . Taking derivatives of (20), we obtain

$$\begin{aligned} \frac{\partial \ell_{ZINB}^c(\Phi; y, z)}{\partial \zeta_j} &= \sum_{i=1}^n \left\{ z_i G_{ij} - \frac{\exp(G_i^T \zeta) G_{ij}}{1 + \exp(G_i^T \zeta)} \right\}, \\ \frac{\partial \ell_{ZINB}^c(\Phi; y, z)}{\partial \beta_k} &= \sum_{i=1}^n \left\{ (1 - z_i) \frac{\partial u_i}{\partial \beta_k} \left( \frac{y_i}{\mu_i} - \frac{y_i + \theta}{\mu_i + \theta} \right) \right\}. \end{aligned} \quad (21)$$

The KKT conditions of an maximizer  $\hat{\Phi}$  for  $\ell_{ZINB}^c(\Phi; y, z)$  can be derived. Therefore, for  $\hat{\zeta}_j = 0, \hat{\beta}_k = 0$ , it must be:

$$\left| \frac{\partial \ell_{ZINB}^c(\Phi; y)}{\partial \zeta_j} \right| \leq n\alpha_1\lambda_1, \quad \left| \frac{\partial \ell_{ZINB}^c(\Phi; y)}{\partial \beta_k} \right| \leq n\alpha_2\lambda_2. \quad (22)$$

The conditional expectation of  $z_i$  at iteration  $m$  is provided by

$$z_i^{(m)} = \begin{cases} \left( 1 + \exp(-G_i^T \zeta^{(m)}) \left[ \frac{\theta}{\exp(B_i^T \beta^{(m)}) + \theta} \right]^\theta \right)^{-1}, & \text{if } y_i = 0 \\ 0, & \text{if } y_i > 0. \end{cases} \quad (23)$$

Let  $\zeta^{(m)} = \zeta, \beta^{(m)} = \beta$ , then (23) becomes

$$z_i = \begin{cases} \left( 1 + \exp(-G_i^T \zeta) \left[ \frac{\theta}{\exp(B_i^T \beta) + \theta} \right]^\theta \right)^{-1}, & \text{if } y_i = 0 \\ 0, & \text{if } y_i > 0. \end{cases} \quad (24)$$

It is simple to show that the right hand side of (21) is the same as that of (16) once (24) is plugged into (21). Hence, the KKT conditions (22) are the same as (18) once (24) is plugged into (21). These connections offer a different method (see Wang et al. (2015)) to derive  $(\hat{\lambda}_{1,\max}, \hat{\lambda}_{2,\max})$  such that  $\hat{\zeta}_j = 0, \hat{\beta}_k = 0$  hold. We first estimate  $\zeta_0, \beta_0$  for an intercept-only ZINB model, then (24) becomes

$$z_i = \begin{cases} \left( 1 + \exp(-\zeta_0) \left[ \frac{\theta}{\exp(\beta_0) + \theta} \right]^\theta \right)^{-1}, & \text{if } y_i = 0 \\ 0, & \text{if } y_i > 0. \end{cases} \quad (25)$$

Plugging in (25),  $(\lambda_{1,\max}, \lambda_{2,\max})$  are computed based on (22). Furthermore, we have shown that  $(\lambda_{1,\max}, \lambda_{2,\max}) = (\hat{\lambda}_{1,\max}, \hat{\lambda}_{2,\max})$  holds.

## References

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